Positivization and regularization of quantum phase space distributions

1994 J. Phys. A: Math. Gen. 27 L39

(http://iopscience.iop.org/0305-4470/27/2/007)

J. Phys. A: Math. Gen. 27 (1994) L39–L45. Printed in the UK

LETTER TO THE EDITOR

Positivization and regularization of quantum phase space distributions

S Chaturvedi, G S Agarwal and V Srinivasan
School of Physics, University of Hyderabad, Hyderabad–500134, India

Received 17 September 1993

Abstract. It is shown that, to any quasiprobability distribution corresponding to a given density operator, one can associate a class of positive distributions in an extended space. In situations where the quasiprobability distributions are singular, these distributions are shown to provide regularized versions thereof. The positive distribution introduced by Drummond and Gardiner in the context of the Glauber–Sudarshan P-representation is shown to arise as a special case. One can thus associate a positive distribution to the Wigner function such that its moments give the averages of Weyl-ordered operators with respect to the given density operator. It is also found that one can associate to the Glauber–Sudarshan P-function, a distribution in the extended space which, though not positive, interestingly, involves the Wigner function. A measurement scheme which directly yields these positive distributions is presented.

Quasiprobability distributions have, over the years, proved to be of immense value in the study of quantum mechanical systems. They are useful not only as computational tools, but can also provide insights into the connections between classical and quantum mechanics. These quasiprobability distributions permit one to express quantum mechanical averages in a form similar to classical averages. The prominent ones among these quasiprobability functions are the Wigner function [1, 2], the Q-function [3] and the P-function of Glauber [4] and Sudarshan [5]. These quasiprobability distributions were introduced with definite objectives in mind and derive their importance in quantum physics from certain specific properties they possess. The Wigner function was originally introduced [1] to study quantum corrections to classical statistical mechanics. Since then, it has found many useful applications in areas such as quantum optics, quantum cosmology, quantum chaos and, more recently, in the context of quantum mechanical histories [6]. Its importance lies in the fact that it plays, in the quantum domain, the role of the classical phase space density and hence provides a bridge between quantum and classical physics. The importance of the Q-function accrues from the fact that, unlike the Wigner function, it has the virtue of being always positive. The Glauber–Sudarshan P-function, in the context of quantum optics, derives its importance from the fact that its moments are directly related to the quantities measured by photodetectors [7]. Each quasiprobability distribution has associated with it a definite rule for operator ordering—the moments of the quasiprobability distributions yield the expectation values of operators ordered according to a specific prescription. Thus, while the moments of the Wigner functions yield expectation values of operators ordered according to the Weyl or symmetric ordering [8], those of the P- and the Q-function, respectively, yield the expectation values of normally and anti-normally ordered operators. A unified treatment of the quasiprobability distributions and the associated ordering prescriptions was given by Agarwal and Wolf [9] and by Cahill and Glauber [10]. In particular, corresponding to a given density operator ρ, Agarwal and Wolf defined a class of quasiprobability distributions $\Phi^{(a)}(\alpha, \alpha^*)$ given by

$$\Phi^{(a)}(\alpha, \alpha^*)=\frac{1}{\pi^2}\int d^2\beta \, \mathrm{Tr}[\rho D(\beta)] \exp(-\tfrac{1}{2}(1-2a)|\beta|^2) \exp(\alpha\beta^*-\alpha^*\beta) \tag{1}$$

where

$$D(\beta)=\exp(\beta\hat{a}^*-\beta^*\hat{a}) \tag{2}$$

and a is a real number $\leqslant 1$. This class of quasiprobability distributions contain the P-function, the Wigner function and the Q-function as special cases—they correspond to the values $a=1$, $1/2$, and 0 respectively. Agarwal and Wolf, in their work [9, P2187, appendix A] also showed that *the quasiprobability distributions* $\Phi^{(a)}(\alpha, \alpha^*)$ *with* $a \leqslant 0$, *for any density operator are always positive.* For the more interesting range of values of a, $1 \geqslant a > 0$ which contains the Wigner and the P-function the positivity of $\Phi^{(a)}(\alpha, \alpha^*)$ is, in general, not guaranteed. The main purpose of this work is to develop a scheme for positivization of these quasiprobability functions.

Positivization by smearing. We begin by noticing that the quasiprobability functions defined in (1) are related to each other by the following formula

$$\Phi^{(a-b)}(\alpha, \alpha^*)=\frac{1}{\pi b}\int d^2\beta \, \Phi^{(a)}(\beta, \beta^*) \exp\left[-\frac{|\alpha-\beta|^2}{b}\right] \tag{3}$$

where $b \geqslant 0$. It is worth noting that for all $b \geqslant a$, $a-b \leqslant 0$ and from the results of Agarwal and Wolf it follows that $\Phi^{(a-b)}(\alpha, \alpha^*)$ is positive. This can, in fact, also be easily proved by noting that the expression multiplying $\Phi^{(a)}(\beta, \beta^*)$ in the RHS of (3) is simply the quasiprobability distribution $\Phi_0^{(1-a)}(\beta, \beta^*)$ corresponding to the density operator ρ' given by

$$\rho'=D(\alpha)\rho_0 D^\dagger(\alpha) \tag{4}$$

where

$$\rho_0=(1-e^{-\theta})\exp(-\theta a^\dagger a) \qquad b-a=(e^\theta-1)^{-1} \tag{5}$$

In view of this, the RHS of (3) is easily seen to be just $\mathrm{Tr}(\rho\rho')$

$$\Phi(a-b)(\alpha, \alpha^*)=\mathrm{Tr}(\rho\rho')=\mathrm{Tr}(\rho D(\alpha)\rho_0 D^\dagger(\alpha)) \tag{6}$$

which, since both ρ and ρ' are positive definite operators, is always positive.

The relation (3) thus suggests a possible scheme of positivization. It associates with any quasiprobability distribution $\Phi^{(a)}(\beta, \beta^*)$, $1 \geqslant a > 0$, a positive distribution $\Phi_{\mathrm{sm}}^{(a)}(\alpha, \alpha^*)$

$$\Phi_{\mathrm{sm}}^{(a)}(\alpha, \alpha^*)=\frac{1}{\pi b}\int d^2\beta \, \Phi^{(a)}(\beta, \beta^*) \exp\left[-\frac{|\alpha-\beta|^2}{b}\right] \tag{7}$$

by smearing the former by the function $(1/\pi b)\exp(-|\alpha-\beta|^2/b)$ with $b > a$. (From this point of view, the quasiprobability distributions $\Phi^{(a)}(\alpha, \alpha^*)$ for $a \leqslant 0$ can be interpreted as smeared versions of $\Phi^{(a)}(\alpha, \alpha^*)$ with a lying in the range $1 \geq a > 0$.) Consider, for instance, the density operators

(1) $\rho = |n\rangle\langle n|$ describing a Fock state,

(2) $\rho = S(z)|0\rangle\langle 0|S^\dagger(z)$; $S(z) = \exp[za^{\dagger 2} - z^* a^2]$, $z = |z|\, e^{i\theta}$ describing a squeezed vacuum, which have highly singular P-functions. The corresponding smeared distributions, respectively, turn out to be

$$P_{sm}(\alpha, \alpha^*) = \frac{1}{b}\left(1 - \frac{1}{b}\right)^n \exp(-|\alpha|^2/b) L_n(-|\alpha|^2/b(b-1)) \qquad b > 1 \qquad (8)$$

and

$$P_{sm}(\alpha, \alpha^*) = [b^2 + (2b-1)\sinh^2|z|]^{-1/2} \times \exp\left[-\frac{\{4(b + \sinh^2|z|)|\alpha|^2 - \sinh^2|z|(\alpha^2 e^{i\theta} + \alpha^{*2} e^{-i\theta})\}}{b^2 + (2b-1)\sinh^2|z|}\right] \qquad (9)$$

which have all the properties of a classical probability distribution.

It is evident from above discussion that this scheme of positivization is amenable to further generalizations. Instead of the thermal density operator ρ_0 in (4), one could use any other suitable density operator to generate a smearing function which would yield a positive distribution $\Phi^{(a)}_{sm}(\alpha, \alpha^*)$ corresponding to a given $\Phi^{(a)}(\alpha, \alpha^*)$. Density operators $\hat{\rho}_0$ of the form $\exp[-$(quadratic form in $\hat{a}^\dagger$ and $\hat{a}$)] are particularly useful for this purpose [6].

This scheme of positivization and regularization by smearing, though widely used in the literature, particularly in the context of the Wigner function, suffers from the drawback that the moments of $\Phi^{(a)}$ and those of the positivized $\Phi^{(a)}_{sm}$ bear no simple relationship. It is thus better to regard the positivized $\Phi^{(a)}_{sm}$ as a new distribution rather than a positivized version of its parent.

Positivization by doubling the number of complex variables. In an important work, Drummond and Gardiner [11] proposed a scheme of positivization of the Glauber–Sudarshan P-function by doubling the number of complex variables. This positive P representation has found numerous applications in quantum optics [11–13]. Drummond and Gardiner show that, to a given density operator ρ, one can associate a manifestly positive distribution $\mathscr{P}(\alpha, \beta, \alpha^*, \beta^*)$ in two complex variables such that its moments $\langle \beta^m, \alpha^n \rangle$ are equal to the moments $\langle \alpha^{*m}\alpha^n \rangle$ of the Glauber–Sudarshan P-function $\Phi^{(1)}(\alpha, \alpha^*)$ corresponding to the given ρ, i.e.

$$\int d^2\alpha (\alpha^*)^m (\alpha)^n P(\alpha, \alpha^*) = \left| d^2\alpha \int d^2\beta (\beta^*)^m (\alpha)^n \mathscr{P}(\alpha, \beta, \alpha^*, \beta^*). \right. \qquad (10)$$

The distribution $\mathscr{P}(\alpha, \beta, \alpha^*, \beta^*)$ is explicitly given by

$$\mathscr{P}(\alpha, \beta, \alpha^*, \beta^*) = \frac{1}{4\pi} \exp[-\tfrac{1}{4}|\alpha - \beta|^2] Q(\tfrac{1}{2}(\alpha + \beta), \tfrac{1}{2}(\alpha^* + \beta^*)). \qquad (11)$$

One can thus associate a positive distribution to a given ρ such that its moments $\langle \beta^m \alpha^n \rangle$ reproduce the averages of the normally ordered operators with respect to ρ.

In this work we show that we can define a positive distribution corresponding to any quasiprobability function $\Phi^{(a)}(\alpha, \alpha^*)$ such that their moments are related as in (10). In other words, corresponding to any operator ordering, we construct a positive distribution in two complex variables such that its moments reproduce the averages, with respect to ρ of operators ordered according to the chosen rule.

Consider the moments of a quasiprobability distribution $\Phi^{(a)}(\alpha, \alpha^*)$

$$\mu_{m,n}^{(a)}=\int d^2\gamma(\gamma^*)^m(\gamma)^n\Phi^{(a)}(\gamma, \gamma^*) \tag{12}$$

where $0\leqslant a\leqslant 1$.

Using the identity

$$(\gamma)^n=\frac{1}{2b\pi}\int d^2\alpha\ \alpha^n \exp\left[-\frac{1}{2b}|\alpha-\gamma|^2\right] \qquad b\geqslant 0 \tag{13}$$

in (12) and changing the order of integration, we obtain

$$\mu_{m,n}^{(a)}=\int d^2\gamma(\gamma^*)^m(\gamma)^n\Phi^{(a)}(\gamma, \gamma^*)=\int d^2\alpha\int d^2\beta(\beta)^m(\alpha)^n\mathscr{P}^{(a,b)}(\alpha, \beta, \alpha^*, \beta^*) \tag{14}$$

where

$$\begin{aligned}\mathscr{P}^{(a,b)}(\alpha, \beta, \alpha^*\ \beta^*)&=\left(\frac{1}{2b\pi}\right)^2\exp\left[-\frac{1}{4b}|\alpha-\beta|^2\right]\int d^2\gamma\Phi^{(a)}(\gamma, \gamma^*)\\&\quad\times\exp\left[-\frac{1}{b}\left|\frac{1}{2}(\alpha+\beta)-\gamma\right|^2\right]\end{aligned} \tag{15}$$

and satisfies the normalization condition

$$\int d^2\alpha\int d^2\beta\ \mathscr{P}^{(a,b)}(\alpha, \beta, \alpha^*, \beta^*)=1. \tag{16}$$

Using the relation (3) we can write (10) as

$$\mathscr{P}^{(a,b)}(\alpha, \beta, \alpha^*, \beta^*)=\frac{1}{4b\pi}\exp\left[-\frac{1}{4b}|\alpha-\beta|^2\right]\Phi^{(a-b)}(\tfrac{1}{2}(\alpha+\beta), \tfrac{1}{2}(\alpha^*+\beta^*)). \tag{17}$$

Thus for a given value of a we have a class of distributions $P^{(a,b)}$ which satisfy (14) and (16) for all values of $b\geqslant 0$. In the following, we show that this freedom in the choice of b can be exploited to associate, to each quasi probability function $\Phi^{(a)}$, a class of manifestly positive distributions $\mathscr{P}^{(a,b)}$ satisfying (14) and (16).

Special cases.

(i) $a=b=1$: Drummond–Gardiner representation

Setting $a=b=1$ in (17), we obtain the Drummond–Gardiner representation for the P-function.

$$\mathscr{P}^{(1,1)}(\alpha, \beta, \alpha^*,\beta^*)=\frac{1}{4\pi}\exp[-\tfrac{1}{4}|\alpha-\beta|^2]Q(\tfrac{1}{2}(\alpha+\beta), \tfrac{1}{2}(\alpha^*+\beta^*)). \tag{18}$$

(ii) $a=b=1/2$: positivized Wigner function
For these values of a and b we obtain the positive distribution

$$\mathscr{P}^{(1/2,1/2)}(\alpha,\beta,\alpha^*,\beta^*)=\frac{1}{2\pi}\exp[-\tfrac{1}{2}|\alpha-\beta|^2]Q(\tfrac{1}{2}(\alpha+\beta),\tfrac{1}{2}(\alpha^*+\beta^*)) \tag{19}$$

which we will, hereafter, refer to as the positivized Wigner function. The moments $\langle\beta^m\alpha^n\rangle$ of this distribution, by construction are equal to the moments $\langle\alpha^{*m}\alpha^n\rangle$ of the Wigner function $W(\alpha,\alpha^*)$.
(iii) $a=0, b=0$
For $a=b=0$ we naturally obtain

$$\mathscr{P}^{(0,0)}(\alpha,\beta,\alpha^*,\beta^*)=\delta^2(\alpha-\beta)Q(\alpha,\alpha^*). \tag{20}$$

(iv) $a=b$
If a is chosen equal to b, we obtain

$$\mathscr{P}^{(a,a)}(\alpha,\beta,\alpha^*,\beta^*)=\frac{1}{4a\pi}\exp\left[-\frac{1}{4a}|\alpha-\beta|^2\right]Q(\tfrac{1}{2}(\alpha+\beta),\tfrac{1}{2}(\alpha^*+\beta^*)). \tag{21}$$

The distributions $\mathscr{P}^{(a,a)}(\alpha,\beta,\alpha^*,\beta^*)$, being equal to the Q-function multiplied by a positive function, are manifestly positive for all values of a. The calculation of $\mathscr{P}^{(a,a)}$ corresponding to a given density operator just involves the computation of the Q-function. The distributions $\mathscr{P}^{(a,a)}$ for the density operator $\rho=|n\rangle\langle n|$. For example, are given by

$$\mathscr{P}^{(a,a)}(\alpha,\beta,\alpha^*,\beta^*)=\frac{2}{4a\pi n!}\left(\frac{|\alpha+\beta|}{2}\right)^{2n}\exp\left[-\frac{1}{4a}|\alpha-\beta|^2-\tfrac{1}{4}|\alpha+\beta|^2\right]. \tag{22}$$

It is worth noting that, while the quasiprobability distributions $\Phi^{(a)}$ themselves are related to each other by (3), the corresponding positive distributions $\mathscr{P}^{(a,a)}$ are related to each other in a rather simple way owing to the fact that the dependence on a appears only through the factor multiplying the Q-function. Thus, $\mathscr{P}^{(1/2,1/2)}$ is related to $\mathscr{P}^{(1,1)}$ as follows

$$\mathscr{P}^{(1/2,1/2)}(\alpha,\beta,\alpha^*,\beta^*)=2\exp[-\tfrac{1}{4}|\alpha-\beta|^2]\mathscr{P}^{(1,1)}(\alpha,\beta,\alpha^*,\beta^*) \tag{23}$$

(v) $b>a$
In this case, since $a-b$ is negative, $\Phi^{(a-b)}$ given by (1) is always positive and so are $\mathscr{P}^{(a,b)}$ for any value of $b>a$. Combining this with the case (iv) we find that, for a given a, we have a class of positive distributions $\mathscr{P}^{(a,b)}$ with $b\geqslant a$.
(vi) $a>b$
In this case (12) associates, to a $\Phi^{(a)}$, a $\mathscr{P}^{(a,b)}$ which involves the quasi probability function $\Phi^{(a-b)}$. Though the $\mathscr{P}^{(a,b)}$ in this case still reproduces the moments of $\Phi^{(a)}$, its positivity is not guaranteed. However it does provide interesting interrelations between quasiprobability distributions. Thus, for instance, for $a=1$ choosing $b=1/2$ one obtains

$$\mathscr{P}^{(1,1/2)}(\alpha,\beta,\alpha^*,\beta^*)=\frac{1}{2\pi}\exp[-\tfrac{1}{2}|\alpha-\beta|^2]W(\tfrac{1}{2}(\alpha+\beta),\tfrac{1}{2}(\alpha^*+\beta^*)). \tag{24}$$

which associates, to the P-function, a quasi-distribution in an extended space which involves the Wigner function.

Measurement of the positive distributions $\mathcal{P}^{(a,b)}(\alpha, \alpha^*, \beta, \beta^*), b \geqslant a$. A scheme for the measurement of the positive distributions $\mathcal{P}^{(a,b)}(\alpha, \alpha^*, \beta, \beta^*)$ can be developed along the lines of that proposed by Arthurs and Kelly [14] and by Braunstein *et al* [15] in the context of the Q-function and the positive P representation respectively. Consider a system interacting with a set of four detectors according to the instantaneous Hamiltonian

$$H_{\text{int}} = \delta(t)[\hat{x}(\hat{p}_1 + \hat{p}_2) + \hat{p}(\hat{p}_3 + \hat{p}_4)] \tag{25}$$

where $\hat{x}, \hat{p}$ and $\hat{x}_i, \hat{p}_i$ denote the position and the momentum operators for the system and the detectors respectively. Given the initial state of the system and the detectors, the quantity of interest is the probability $P(x_1, x_2, x_3, x_4)$ that the detector coordinates take values x_1, x_2, x_3 and x_4 after the detectors have interacted with the system. This joint probability, by definition, is given by

$$P(x_1, x_2, x_3, x_4) = \langle U^\dagger \delta(\hat{x}_1 - x_1)\delta(\hat{x}_2 - x_2)\delta(\hat{x}_3 - x_3)\delta(\hat{x}_4 - x_4) U \rangle \tag{26}$$

where (with $\hbar = 1$ hereafter)

$$U = \exp(-\mathrm{i}[\hat{x}(\hat{p}_1 + \hat{p}_2) + \hat{p}(\hat{p}_3 + \hat{p}_4)]) \tag{27}$$

and $\langle \ldots \rangle$ denotes the average with respect to the initial density operator $\rho(0)$.

We further assume that initially the system and the detector are uncorrelated so that $\rho(0) = \rho_s \rho_d$ and that $\rho_d = |\psi\rangle\langle\psi|$ with

$$\langle X_1, X_2, X_3, X_4 | \psi \rangle = \frac{2}{\pi b} \exp\left[-\frac{1}{b}(X_1^2 + X_3^2 + X_2^2 + X_4^2) \right]. \tag{28}$$

Here $X_1 = (x_1 + x_2)/2, X_3 = (x_3 + x_4)/2, X_2 = (x_1 - x_2)/2$ and $X_4 = (x_3 - x_4)/2$. X_1 and X_3 can be looked upon as the centre-of-mass coordinates and X_2 and X_4 as the relative coordinates. In terms of the position variables (28) implies that the initial wavefunction of the detector is chosen to be

$$\langle x_1, x_2, x_3, x_4 | \psi \rangle = \frac{1}{\pi b} \exp\left[-\frac{1}{2b}(x_1^2 + x_2^2 + x_3^2 + x_4^2) \right]. \tag{29}$$

We further assume that $b \leqslant 1$ i.e. all the detectors are identically prepared in a squeezed state characterized by the parameter b. With these assumptions regarding the initial state of the detectors, a simple computation shows that

$$P(x_1, x_2, x_3, x_4) = \mathcal{P}^{(a,b)}(\alpha, \beta, \alpha^*, \beta^*) \tag{30}$$

where

$$(\alpha + \beta) = \sqrt{2}(X_1 + \mathrm{i}X_3) \qquad (\alpha - \beta) = 2\sqrt{2}(X_2 + \mathrm{i}X_4) \tag{31}$$

and

$$a = \frac{1}{4}\left[2 + 3b - \frac{1}{b} \right]. \tag{32}$$

The situation considered by Braunstein *et al* [15] corresponds to $b = 1$. Thus we see that, if the detector is initially prepared in the state (29) the statistics of the four detectors directly gives the positive distribution $\mathcal{P}^{(a,b)}(\alpha, \beta, \alpha^*, \beta^*)$ corresponding to the initial density operator of the system. If b is chosen to lie in the range $1 \geqslant b \geqslant 1/3$, the outcome of the above measurement would be the distribution $\mathscr{P}^{(a,b)}(\alpha, \beta, \alpha^*, \beta^*)$ with a lying in the range $1 \geqslant a \geqslant 0$. Thus the choices $b = 1/3$, $1/\sqrt{3}$, and 1 yield the distributions $\mathscr{P}^{(0,1/3)}$, $\mathscr{P}^{(1/2,1/\sqrt{3})}$ and $\mathscr{P}^{(1,1)}$ respectively.

To conclude, we have shown how a class of quasiprobability functions may be positivized. Our work may be viewed as a generalization of that of Drummond and Gardiner on positivization of the Glauber–Sudarshan P-function. The class of quasiprobability functions considered here includes the Wigner function as well. In contrast to the other positivization schemes such as smearing, the moments of these positive distributions are directly related to those of the quasiprobability distributions and hence to the averages of operators ordered according to a specific prescription. The positive distributions constructed in the present work also provide a way to regularize the quasiprobability distributions when the latter become singular. A measurement scheme which directly yields some of these positive distributions is also presented.

SC is grateful to Professor C L Mehta and Ms P Shanta for discussions

References

[1] Wigner E P 1932 *Phys. Rev.* **40** 749
[2] Hillery M, O'Connell R F, Scully M O and Wigner E P 1984 *Phys. Rep.* **106** 121
[3] Kano Y 1964 *J. Phys. Soc. Japan* **19** 1555
Mehta C L and Sudarshan E C G 1965 *Phys. Rev.* **138** 274
[4] Glauber R J 1963 *Phys. Rev. Lett.* **10** 84
[5] Sudarshan E C G 1963 *Phys. Rev. Lett.* **10** 277
[6] Halliwell J J 1992 *Phys. Rev.* D **46** 1610
[7] Klauder J R and Sudarshan E C G 1968 *Fundamentals of Quantum Optics* (New York: Benjamin)
[8] Weyl H 1931 *Theory of Groups and Quantum Mechanics* (New York: Dover)
[9] Agarwal G S and Wolf E 1970 *Phys. Rev.* D **2** 2161, 2187, 2206
[10] Cahill K E and Glauber R J 1969a *Phys. Rev.* **177** 1857, 1882
[11] Drummond P D and Gardiner C W 1980 *J. Phys. A: Math. Gen.* **13** 2353
[12] Drummond P D, Gardiner C W and Walls D F 1981 *Phys. Rev.* A **24** 914
[13] Gardiner C W 1983 *Handbook of Stochastic Methods* (Berlin: Springer) p 408
[14] Arthurs E and Kelly J L Jr 1965 *Bell. Syst. Tech. J.* **44** 725
[15] Braunstein S L, Caves C M and Milburn G J 1991 *Phys. Rev.* A **43** 1153